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How majorities can lose the election Another voting paradox

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Abstract. We show that due to free riding of potential voters facing positive voting costs, the proposal with the highest number of supporters can still be the most likely to lose a binary election.

1 Introduction

Ever since Condorcet (1785) and De Borda (1781), social choice theorists have been concerned with voting paradoxes. Usually, such paradoxes consider three or more alternatives, preference orderings of voters over these alternatives, and show that some voting procedure yields an aggregate preference ordering that is in some sense inconsistent with the individual preference orderings. For a survey, see Nurmi (1998), who loosely defines that "a voting paradox occurs whenever the relationship between the voting result and the voter preferences is counter-intuitive or unreasonable in some sense" (p. 335).

In this paper, we consider a paradox that occurs in an even simpler set-up. Suppose we have a binary election, where the only alternatives are A and B. The number of supporters for each alternative is common knowledge. Alternative B has only one supporter, while A has $n > 1$. There are costs to voting, and there is a possibility to abstain. We then show that proposal B can have a higher probability of winning the election, *even though the number of supporters of proposal A is higher*. The intuition is that the higher the number of supporters, the higher the incentive of each supporter to free ride on the vote of the other supporters, and save the costs of voting. When these costs are high, the free-riding effect can be so strong, that the fact that A has more sup-

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porters, is outweighed. When there is more than 1 supporter for B, the results become less clear cut. Supporters of B now also have an incentive to free ride. Yet we show that, also in that case, there is a range of voting costs such that a symmetric equilibrium exists in which the probability that B wins is higher, even though A has more supporters.

Some recent papers are related to our model. Myerson (2000) studies elections with a similar set-up to those that we are studying. In his model, the number of potential voters is large but uncertain, and the preferred policy position and voting costs of each of them are given by draws from given probability distributions. In Osborne et al. (2000) people choose whether or not to attend a meeting, rather than whether or not to vote in an election. The outcome of the meeting is a function of the preferences of the people that show up. In their model, there is a continuum of possible policy preferences. Also, the proposal that is ultimately decided upon, is dependent on the people that show up. In our model, people have one of two possible preferences: they either support or are opposed to a proposal that is already known before they decide whether or not to vote.

2 A voting model

Suppose there are $n+1$ potential voters. Out of these, n are known to be a supporter of proposal A. The other is a supporter of proposal B. As a normalization, each potential voter obtains utility 2 when his favorite proposal wins the election, and 0 when the other proposal wins. If there is a draw, the winner is decided by coin toss, and every player obtains expected utility 1. Each potential voter can choose between two pure strategies: vote for his favorite proposal, or abstain. Of course, a potential voter may choose to vote against his favorite proposal, but it is easy to see that this is a dominated strategy. The costs of voting are c, equal for all players. For a non-trivial problem, we require[1] $0 < c < 1$.

The full set of supporters of A is denoted N. The set of supporters of A other than some i then is $N \backslash \{i\}$. The set of supporters of B is denoted M. The set of supporters of B other than some i is then $M \backslash \{i\}$. In this section, M only has one element, so we simply have $M \backslash \{i\} = \emptyset$. We write $P(X|Y, Z)$ for the probability that event X occurs, given that the set of supporters of A is Y and the set of supporters of B is Z. In this context, we denote $\#A$ the number of supporters of A in subset Y that votes, and $\#B$ the number of supporters of B in subset Z that votes.

A pure strategy equilibrium does not exist.[2] Consider the possibility of a mixed strategy equilibrium. By abstaining, supporter i of proposal A obtains

[1] Since any voter can only change the outcome from either a loss to a draw, or from a draw to a win, the expected benefit of voting cannot exceed 1. Thus, we need $c < 1$ to have any turnout in equilibrium.
[2] See Appendix, Lemma 1.

utility 2 if the number of voters among the other supporters of A turns out to be higher than the number of voters for proposal B. He obtains 1 if these numbers turn out to be equal. Thus, his expected utility of abstaining equals

$$U_i(\text{abstain}) = 2 \cdot P(\#A > \#B \mid N \backslash \{i\}, M) + P(\#A = \#B \mid N \backslash \{i\}, M). \quad (1)$$

When voting, his expected utility is

$$U_i(\text{vote}) = 2 \cdot P(\#A \geq \#B \mid N \backslash \{i\}, M) + P(\#A + 1 = \#B \mid N \backslash \{i\}, M) - c.$$

Necessary for a mixed strategy equilibrium is that he is indifferent, thus

$$P(\#A = \#B \mid N \backslash \{i\}, M) + P(\#A + 1 = \#B \mid N \backslash \{i\}, 1) = c. \quad (2)$$

Note that the LHS of this equality gives the expected gross benefit of voting for i, while the RHS is the cost of doing so. If the LHS is larger than c, it is a unique best response to vote. If it is smaller, it is a unique best response to abstain. Similarly, a supporter of position B is indifferent if

$$P(\#A = \#B \mid N, M \backslash \{i\}) + P(\#A = \#B + 1 \mid N, M \backslash \{i\}) = c. \quad (3)$$

As usual, we focus on Nash equilibria that are symmetric, in the sense that potential voters with the same preference use the same strategy. At the end of this section, we also look at asymmetric equilibria. Let P_A be the probability that a given supporter of A votes, and P_B the probability that a given supporter of B votes. Note that for this case

$$\begin{aligned} P(\#A = \#B \mid N \backslash \{i\}, M) &= (1 - P_A)^{n-1}(1 - P_B) \\ &\quad + (n-1)P_A(1 - P_A)^{n-2}P_B, \\ P(\#A + 1 = \#B \mid N \backslash \{i\}, M) &= (1 - P_A)^{n-1}P_B, \\ P(\#A = \#B \mid N, M \backslash \{i\}) &= (1 - P_A)^n, \\ P(\#A = \#B + 1 \mid N, M \backslash \{i\}) &= nP_A(1 - P_A)^{n-1}. \end{aligned} \quad (4)$$

Plugging this into (2) and (3) yields

$$(1 - P_A^*)^{n-1} + (n-1)P_A^*(1 - P_A^*)^{n-2}P_B^* = c, \quad (5)$$

$$(1 - P_A^*)^n + nP_A^*(1 - P_A^*)^{n-1} = c, \quad (6)$$

where asterisks denote equilibrium values. Equating the LHSs of these equations yields

$$(1 - P_A^*) + (n-1)P_A^*P_B^* = (1 - P_A^*)^2 + nP_A^*(1 - P_A^*),$$

thus

$$(n-1)P_A^*P_B^* + P_A^*(1 - P_A^*) = nP_A^*(1 - P_A^*),$$

hence

$$P_B^* = 1 - P_A^*. \quad (7)$$

From (6) we can study the effect of an increase in the cost of voting c on the equilibrium probability that any of the n supporters of A will vote, which is P_A^*. Note

$$\frac{\partial}{\partial P_A^*}((1-P_A^*)^n + nP_A^*(1-P_A^*)^{n-1}) = -n(n-1)P_A^*(1-P_A^*)^{n-2} < 0 \qquad (8)$$

for $0 < P_A^* < 1$ and $n > 1$. Thus, from (6), the probability that any supporter of A votes, is decreasing in c. Also note

$$\lim_{P_A^* \to 0}((1-P_A^*)^n + nP_A^*(1-P_A^*)^{n-1}) = 1.$$

Combined with the previous result, this implies that the probability that any supporter of A votes, goes to zero as c approaches 1. The probability that B wins equals

$$P(B\,\text{wins}) = P_B^*(1-P_A^*)^n.$$

Using (7) this yields

$$P(B\,\text{wins}) = (1-P_A^*)^{n+1},$$

which is decreasing in P_A^*. This expression approaches 1 as P_A^* approaches 0. But that implies that as c goes to 1, the probability that B wins approaches 1. Obviously, the opposite holds for A. Using (7), the probability that A wins equals

$$P(A\,\text{wins}) = 1-(1-P_A^*)^n - nP_A^*(1-P_A^*)^n,$$

which approaches 0 as P_A^* goes to zero. We have now proven the following

Proposition 1. *With the number of supporters of A equal to $n > 1$, and only 1 supporter of B, there is a c^* such that the probability that proposal B wins is higher than the probability that A wins for any $c > c^*$. The probability that B wins goes to 1 as c goes to 1.*

As an example, consider the case in which there are 2 supporters for proposal A and 1 supporter for B. Then

$$P(A\,\text{wins}) = 1-(1-P_A^*)^2 - 2P_A^*(1-P_A^*)^2,$$

$$P(B\,\text{wins}) = (1-P_A^*)^3,$$

thus $P(B\,\text{wins}) > P(A\,\text{wins})$ iff $P_A^* < 0.347$. From (6), we now have

$$(1-P_A^*)^2 + 2P_A^*(1-P_A^*) = c,$$

thus $P_A^* = \sqrt{(1-c)}$ and $P_B^* = 1-\sqrt{(1-c)}$. This implies that $P(B\,\text{wins}) > P(A\,\text{wins})$ iff $c > 0.72193$.

Arguably, the situation described in Proposition 1 can be classified as a voting paradox. Even though the number of supporters for proposal A is higher, proposal B more often wins the election.[3] The explanation for this paradox is straightforward. The one supporter of position B knows that he cannot win the election when he does not vote, giving him a strong incentive to do so. Yet, supporters of position A have an incentive to free ride on the efforts of the other supporters. Doing so saves them the cost of voting. Two effects ultimately decide which proposal has a better chance to win. On the one hand, there is a numbers effect: when the probability of voting is equal for all players, A has a higher probability of winning since the number of supporters of A is higher. On the other hand, there is a free riding effect. The fact that A has more supporters gives each supporter a higher incentive to free ride. As the cost of voting c increases, the free riding effect becomes stronger. For high enough c, the free riding effect dominates the numbers effect, and B has a higher probability of winning.

In the Appendix we also look at the asymmetric equilibria of this game. Define P_i^* as the equilibrium probability that supporter i of A votes, $i \in N$. We show that the full set of equilibria can be characterized as follows

Proposition 2. *With n supporters of A and 1 supporter of B, equilibria in which all supporters vote with positive probability are given by*

1. $P_i^* = P_A^* \; \forall i \in N$ *and* $P_B = 1 - P_A^*$, *with* P_A^* *implicitly defined by (6),*
2. $P_i^* = 1$ *for some* $i \in N$, $P_j^* = \sqrt[n-1]{c} \; \forall j \in N \backslash \{i\}$, *and* $P_B^* = 1 - \sqrt[n-1]{c}$.

Moreover, we have equilibria in which a supporters of A abstain, whereas all other supporters play one of the equilibria defined above, for a game where A has n − a supporters.

Thus, there is a unique equilibrium in which all voters use a mixed strategy. Hence, the symmetric equilibrium is also unique. Moreover, there is a set of equilibria in which one supporter of A votes with certainty and all others follow a mixed strategy. Obviously, in those equilibria, the probability that A wins is always higher than the probability that B wins. Finally, given any equilibrium in a game with $n - a$ supporters of A, we can obtain an equilibrium in a game with n supporters of A by simply adding a supporters that abstain with certainty. Naturally, adding those abstainers does not influence the equilibrium probability of winning of either proposal.

Hence, when looking at asymmetric equilibria, our voting paradox does not always occur. This is intuitive. In equilibria without a paradox, one supporter of A votes with certainty. Effectively, in these equilibria, the supporters of A coordinate on one of them to bear the bulk of the voting costs necessary to avoid losing the election. When such coordination is possible, free riding is no longer an issue, and the paradox is avoided.

[3] One may even argue that we already have a paradox when the probability that A wins is strictly smaller than 1. Then, it is easy to see that we have a voting paradox for any c. Since only mixed strategy equilibria exist, there is always some probability that A does not win the election.

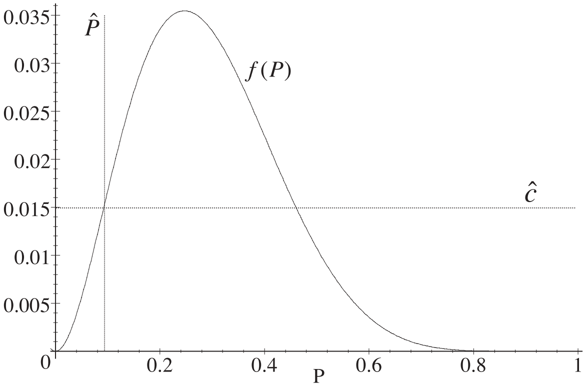

Fig. 1. Deriving the equilibrium when B has more than 1 supporter

3 More supporters for *B*

We now consider the case in which there is more than one supporter of B, but the number of supporters of A is still higher. This greatly complicates the analysis. Now, supporters of B also have an incentive to free ride on each other's efforts, and we have multiple symmetric equilibria. Yet, in the appendix we show that, for a range of voting costs c, there is a symmetric equilibrium in which our voting paradox occurs:

Proposition 3. *Consider the game with n suppporters of A and m supporters of B, with $n > m > 1$. For small enough c, there exists a symmetric equilibrium in which the probability that B wins is larger than the probability that A wins. In this equilibrium, $P_A^* = 1 - P_B^* = P$, with P the smallest positive number satisfying*

$$P^m(1-P)^{n-1}n + P^{m-1}(1-P)^n m = \frac{n!m!}{(n+m-1)!}c. \tag{9}$$

The argument is illustrated in Fig. 1. In the Appendix, we show that we have a symmetric equilibrium whenever (9) is satisfied. For the case $n = 8$ and $m = 3$, the horizontal line in Fig. 1 represents the RHS of (9), and the curve gives the LHS. The Appendix shows that this curve is hump-shaped for any $n > m > 1$. Since $P_A = P$ and $P_B = 1 - P$, the probability that B wins the election will be larger than the probability that A wins if $P < \hat{P}$, with $\hat{P}$ some critical value. From the figure, it is clear that if c is small enough, that is, if the RHS of (9) is smaller than some $\hat{c}$ corresponding to $\hat{P}$, we have an equilibrium with $P < \hat{P}$, so the probability that B wins is larger than the probability that A wins.

Interestingly, the paradox now occurs for low voting costs rather than for high voting costs, as we had with $m = 1$. For high c, a symmetric equilibrium may not exist, at least not of the form $P_A = 1 - P_B$. Also note that the symmetric equilibrium is not unique. From the figure, for any $c < \hat{c}$, there is also an equilibrium with $P > \hat{P}$, i.e. where the probability that A wins is higher. On the one hand, our paradox is therefore less pervasive than in the case with $m = 1$, since the paradox equilibrium is no longer the unique symmetric equilibrium. But on the other hand, the paradox is more pervasive, since it may now occur for small rather than large c. This implies that, even if the importance of the election outcome is high relative to the voting costs, we still may end up in an equilibrium where the minority wins the election.

4 Conclusion

In this paper, we identified another voting paradox. In a binary election where potential voters can abstain and there is a cost of voting, the proposal with the lowest support may still be the most likely to win the election. When the minority only has one member, there is a unique symmetric equilibrium. For high enough voting costs, in this equilibrium the probability that the minority wins is higher than the probability that the majority wins. Members of the majority have an incentive to free ride on each other, giving the minority an advantage. When the minority has more than one member, the analysis is less clear-cut. Members of the minority now also have an incentive to free ride, and there are multiple symmetric equilibria. Yet, in at least one of those, the probability that the minority wins is higher if voting costs are low enough.

Appendix

A.1 Proof of proposition 2

Throughout the following analysis, we will refer to the one supporter of B simply as B. For expositional convenience, we also suppress the asterisk superscripts on equilibrium probabilities. We first show the following result

Lemma 1. *There cannot be an equilibrium in which all supporters of A use a pure strategy.*

Proof. Suppose that there is. First suppose the equilibrium number of voters for A is zero. Then the best response for B is to vote, since this yields net benefit $1 - c$. But then it is a best response for any supporter of A also to vote, destroying this possibility. Next, consider the possibility of such an equilibrium with $\#A = 1$. The best response for B is to vote. But then it is the best response for any of the non-voting supporters of A also to vote, destroying this possibility as well. Finally, consider the possibility of such an equilibrium with $\#A > 1$. The best response for B is to abstain. But then the best response of any of the voting supporters of A is to abstain as well, also destroying this possibility and establishing the proof. //

Next, we define a strictly mixed strategy equilibrium as an equilibrium in which *all* voters play a mixed strategy. We can show

Lemma 2. *All strictly mixed strategy equilibria are symmetric.*

Proof. Denote the probability that supporter i of A votes as P_i, with $i \in N$. The probability that B votes is P_B. First, assume $n > 2$. In this context

$$\begin{aligned} P(\#A = \#B \mid N \backslash \{i\}, M) &= (1 - P_B) \prod_{j \neq i} (1 - P_j) \\ &\quad + P_B \sum_{j \neq i} P_j \prod_{k \neq i, j} (1 - P_k), \\ P(\#A + 1 = \#B \mid N \backslash \{i\}, M) &= P_B \prod_{j \neq i} (1 - P_j), \\ P(\#A = \#B \mid N, M \backslash \{i\}) &= \prod_j (1 - P_j), \\ P(\#A = \#B + 1 \mid N, M \backslash \{i\}) &= \sum_j P_j \prod_{k \neq j} (1 - P_k). \end{aligned}$$

so (2) reduces to

$$\prod_{j \neq i} (1 - P_j) + P_B \sum_{j \neq i} P_j \prod_{k \neq i, j} (1 - P_k) = c \quad \forall i \in N, \tag{9'}$$

whereas (3) implies

$$\prod_j (1 - P_j) + \sum_j P_j \prod_{k \neq j} (1 - P_k) = c. \tag{10}$$

Consider two different supporters of A, r and s, with $r \neq s$. For r, condition (9′) can be written

$$\begin{aligned} &(1 - P_s) \prod_{j \neq r, s} (1 - P_j) \\ &\quad + P_B \left(P_s \prod_{k \neq r, s} (1 - P_k) + (1 - P_s) \sum_{j \neq r, s} P_j \prod_{k \neq r, s, j} (1 - P_k) \right) = c, \end{aligned}$$

thus

$$(1 - P_s + P_B P_s) \prod_{j \neq r, s} (1 - P_j) + P_B (1 - P_s) \sum_{j \neq r, s} P_j \prod_{k \neq r, s, j} (1 - P_k) = c.$$

Since we assume that all P_is are strictly between 0 and 1, this can be rewritten

$$(1 - P_s + P_B P_s) \prod_{j \neq r, s} (1 - P_j) + P_B (1 - P_s) \sum_{j \neq r, s} \frac{P_j}{1 - P_j} \prod_{k \neq r, s} (1 - P_k) = c,$$

or

$$(1 - P_s + P_B P_s) \prod_{j \neq r, s} (1 - P_j) + P_B (1 - P_s) \sum_{k \neq r, s} \frac{P_k}{1 - P_k} \prod_{j \neq r, s} (1 - P_j) = c,$$

hence

$$\left(\prod_{j\neq r,s}(1-P_j)\right)\left(1-P_s+P_BP_s+P_B(1-P_s)\sum_{k\neq r,s}\frac{P_k}{1-P_k}\right)=c, \tag{11}$$

In the same way, the condition for supporter s can be written

$$\left(\prod_{j\neq r,s}(1-P_j)\right)\left(1-P_r+P_BP_r+P_B(1-P_r)\sum_{k\neq r,s}\frac{P_k}{1-P_k}\right)=c, \tag{12}$$

Equating the left hand sides of (11) and (12) yields

$$-P_s(1-P_B)+P_B(1-P_s)\sum_{k\neq r,s}\frac{P_k}{1-P_k}$$
$$=-P_r(1-P_B)+P_B(1-P_r)\sum_{k\neq r,s}\frac{P_k}{1-P_k},$$

hence

$$(P_r-P_s)\left(1-P_B+P_B\sum_{k\neq r,s}\frac{P_k}{1-P_k}\right)=0.$$

The second bracketed term is strictly positive for any $P_B\in(0,1)$ and $P_k\in(0,1)$ $\forall k\neq r,s$. Thus we necessarily have $P_r=P_s$ for all r,s. Hence, an asymmetric mixed strategy equilibrium does not exist. For the case $n=2$, conditions (2) reduce to $1-P_2(1-P_B)=c$ and $1-P_1(1-P_B)=c$, which, given that $P_B<1$, also directly implies that $P_1=P_2$. //

We immediately have

Lemma 3. *There is a unique strictly mixed strategy equilibrium.*

Proof. From the previous lemma, any strictly mixed strategy equilibrium is symmetric. From the derivation in the main text, using (5) and (7), such an equilibrium can be characterized by the probability P_A^* that a given supporter of A will vote, where we necessarily have

$$(1-P_A^*)^n+nP_A^*(1-P_A^*)^{n-1}=c. \tag{13}$$

For $P_A^*=1$, the LHS equals 0. For $P_A^*=0$, it equals 1. From (8), the LHS is strictly decreasing. Hence, the equality has a unique solution. //

We now consider whether there can be equilibria in which some voters use a pure strategy, whereas others use a mixed one. It is easy to see that

Lemma 4. *There cannot be more than 1 supporter of A that votes with certainty.*

Proof. Suppose there is more than one. The best response for all others then is to abstain. But then the best response for either of the two voters is to abstain as well. //

Now consider whether there can be one supporter of A that votes with certainty, i.e., whether some P_j can equal 1. We have

Lemma 5. *There is a set of equilibria in which 1 supporter of A votes with certainty, all other supporters of A vote with probability* $\sqrt[n-1]{c}$, *while the supporter of B votes with probability* $1 - \sqrt[n-1]{c}$. *These are the only equilibria in which one supporter of A votes with certainty, while all other supporters vote with positive probability.*

Proof. Suppose without loss of generality that $P_1 = 1$. We look for all equilibria in which the other supporters vote with positive probability. From the previous lemma, $P_i \neq 1 \ \forall i > 1$. Thus all other supporters of A must be indifferent between voting and not voting. Condition $(9')$ then implies

$$P_B \prod_{k \neq 1, i} (1 - P_k) = c, \quad \forall i \in N \backslash \{1\}.$$

whereas (10) implies

$$\prod_{k \neq 1} (1 - P_k) = c,$$

thus $P_B = 1 - P_i$, $\forall i > 1$. Plugging that back into either of the above equalities yields

$$(1 - P_i)^{n-1} = c, \tag{14}$$

hence $P_i = \sqrt[n-1]{c}$ and $P_B = 1 - \sqrt[n-1]{c}$. From $(9')$, for supporter 1 to indeed be willing to vote with certainty, we need

$$\prod_{j \neq 1} (1 - P_j) + P_B \sum_{j \neq 1} P_j \prod_{k \neq 1, j} (1 - P_k) \geq c.$$

hence $(1 - P)^{n-1} + (n-1)P(1 - P)^{n-1} \geq c$, thus $(1 - P)^{n-1}(1 + nP - P) \geq c$. Combined with (14) this requires $1 + nP - P \geq 1$, thus $P(n-1) \geq 0$. This is always satisfied, which established the result for $n > 2$. It is easy to verify that for $n = 2$, the lemma is also satisfied. //

Lemma 6. *There is no equilibrium in which B votes with certainty.*

Proof. Suppose $P_B = 1$. From lemma 5, if one supporter of A votes with certainty, then B cannot vote with certainty. From lemma 4, we have that there can never be more than 1 supporter of A that votes with certainty. Hence, with $P_B = 1$, no supporter of A votes with certainty. Consider the subset $\mathcal{N}$ of supporters of A that vote with positive probability. Lemma 1 implies that such a subset exists. First, assume $\#\mathcal{N} > 2$. From $(9')$, for these supporters we need

$$\prod_{j \in \mathcal{N} \backslash \{i\}} (1 - P_j) + \sum_{j \in \mathcal{N} \backslash \{i\}} P_j \prod_{k \in \mathcal{N} \backslash \{i, j\}} (1 - P_k) = c \quad \forall i \in \mathcal{N}, \tag{15}$$

whereas for B to vote with certainty, we need from (10)

$$\prod_{j\in\mathcal{N}}(1-P_j)+\sum_{j\in\mathcal{N}}P_j\prod_{k\in\mathcal{N}\setminus\{j\}}(1-P_k)\geq c,$$

thus for all $i\in\mathcal{N}$,

$$(1-P_i)\prod_{j\in\mathcal{N}\setminus\{i\}}(1-P_j)+P_i\prod_{k\in\mathcal{N}\setminus\{i\}}(1-P_k)$$
$$+(1-P_i)\sum_{j\in\mathcal{N}\setminus\{i\}}P_j\prod_{k\in\mathcal{N}\setminus\{i,j\}}(1-P_k)\geq c,$$

Using (15), we then have $(1-P_i)c+P_i\prod_{k\in\mathcal{N}\setminus\{i\}}(1-P_k)\geq c$, or $\prod_{j\in\mathcal{N}\setminus\{i\}}(1-P_j)\geq c$, which clearly violates (15), thus establishing the result for $\#\mathcal{N}>2$. For $\#\mathcal{N}=2$, (2) again implies $1-P_i(1-P_B)=c\ \forall i\in\mathcal{N}$. With $P_B=1$, this is never satisfied. Finally, note that $\#\mathcal{N}=1$ is not feasible, since with B voting and all other supporters of A abstaining, it is a unique best response for the one member of $\mathcal{N}$ to vote as well. //

Lemma 7. *There is no equilibrium in which B abstains with certainty.*

Proof. Suppose $P_B=0$. This implies that no supporter i of A can have $P_i=1$. Consider the subset $\mathcal{N}$ of supporters of A that vote with positive probability. First, consider the possibility $\#\mathcal{N}>1$. From (9′), they have

$$\prod_{j\in\mathcal{N}\setminus\{i\}}(1-P_j)=c\quad\forall i\in\mathcal{N}.\tag{16}$$

For B to be willing not to vote, we need, from (10), $\prod_{j\in\mathcal{N}}(1-P_j)+\sum_{j\in\mathcal{N}}P_j\prod_{k\in\mathcal{N}\setminus\{j\}}(1-P_k)\leq c$, or, from (16), $(1-P_i)c+c\sum_{j\in\mathcal{N}}P_j\leq c$. Hence, we need $\sum_{j\in\mathcal{N}}P_j\leq P_i$. Since the sommation on the LHS also includes P_i, this can not be satisfied, thus establishing the result for $\mathcal{N}>1$. Finally, note that $\#\mathcal{N}=1$ is not feasible, since with B and all other supporters of A abstaining, it is a unique best response for the one member of $\mathcal{N}$ to vote as well. //

Lemma 8. *If $\{P_1,\ldots,P_n;P_B\}$ is an equilibrium in the game with n supporters for A, then $\{P_1,\ldots,P_n,0;P_B\}$ is an equilibrium in the game with $n+1$ supporters.*[4]

Proof. Consider an equilibrium in a game with N supporters of A, that is given by $\{P_1,\ldots,P_n,P_B\}$. From the previous results we know that, in this equilibrium, B is indifferent between voting and abstaining while at least one supporter of A also is. For B, (10) is satisfied, while for the indifferent supporter of A, (9′) is. Consider that, given this equilibrium, another supporter of A enters the game. Along similar lines as in the derivation of (9′), his expected revenue from voting is

$$\prod_{j\in\mathcal{N}}(1-P_j)+P_B\sum_{j\in\mathcal{N}}P_j\prod_{k\in\mathcal{N}\setminus\{j\}}(1-P_k).$$

[4] We thank an anonymous referee for pointing out this possibility.

But from (10),

$$\prod_{j\in\mathcal{N}}(1-P_j)+\sum_{j\in\mathcal{N}}P_j\prod_{k\in\mathcal{N}\setminus\{j\}}(1-P_k)=c.$$

Since in any equilibrium $P_B < 1$, the net benefit of voting for the additional supporter is strictly smaller than c. Hence, his unique best reply is to abstain. Observing that the presence of additional abstaining supporters does not affect the incentives of the others, we have established the proof. //

Collecting all the results derived above, we now have established Proposition 2.

A.2 Proof of proposition 3

Now consider the general case with n supporters of A, and m supporters of B, with $n > m$. Thus, the set M now contains more than one element. We restrict attention to symmetric equilibria. In this context

$$\begin{aligned}
&P(\#A=\#B\,|\,N\setminus\{i\},M)\\
&\quad=\sum_{j=0}^{m}\binom{n-1}{j}P_A^j(1-P_A)^{n-1-j}\binom{m}{j}P_B^j(1-P_B)^{m-j},\\
&P(\#A+1=\#B\,|\,N\setminus\{i\},M)\\
&\quad=\sum_{j=0}^{m-1}\binom{n-1}{j}P_A^j(1-P_A)^{n-1-j}\binom{m}{j+1}P_B^{j+1}(1-P_B)^{m-j-1},\\
&P(\#A=\#B\,|\,N,M\setminus\{i\})\\
&\quad=\sum_{j=0}^{m-1}\binom{n}{j}P_A^j(1-P_A)^{n-j}\binom{m-1}{j}P_B^j(1-P_B)^{m-j-1},\\
&P(\#A=\#B+1\,|\,N,M\setminus\{i\})\\
&\quad=\sum_{j=0}^{m-1}\binom{n}{j+1}P_A^{j+1}(1-P_A)^{n-1-j}\binom{m-1}{j}P_B^j(1-P_B)^{m-j-1},
\end{aligned}\tag{17}$$

with P_k the probability that any given supporter of k will vote, $k \in \{A, B\}$. A necessary condition for a strictly mixed strategy equilibrium is for the LHS of (2) and (3) to be equal, i.e. for the expected benefit of voting to be equal for all potential voters. The problem of finding an equilibrium can be simplified by observing the following:

Lemma 9. *Sufficient for the expected benefit of voting to be equal for all potential voters is*

$$P_B = 1 - P_A.$$

Proof. Assume $P = P_A = 1 - P_B$. Using (2), (3), and (17), expected benefits are equal if

$$P^m(1-P)^{n-1}\sum_{j=0}^{m}\binom{n-1}{j}\binom{m}{j} + P^{m-1}(1-P)^n\sum_{j=0}^{m-1}\binom{n-1}{j}\binom{m}{j+1} = P^{m-1}(1-P)^n\sum_{j=0}^{m-1}\binom{n}{j}\binom{m-1}{j} + P^m(1-P)^{n-1}\sum_{j=0}^{m-1}\binom{n}{j+1}\binom{m-1}{j},$$

which reduces to

$$P\sum_{j=0}^{m}\binom{n-1}{j}\binom{m}{j} + (1-P)\sum_{j=0}^{m-1}\binom{n-1}{j}\binom{m}{j+1} = (1-P)\sum_{j=0}^{m-1}\binom{n}{j}\binom{m-1}{j} + P\sum_{j=0}^{m-1}\binom{n}{j+1}\binom{m-1}{j},$$

hence

$$P\binom{n+m-1}{m} + (1-P)\binom{n+m-1}{n} = (1-P)\binom{n+m-1}{n} + P\binom{n+m-1}{m},$$

which is always satisfied.

To solve for P, we have from (2)

$$P^m(1-P)^{n-1}\binom{n+m-1}{m} + P^{m-1}(1-P)^n\binom{n+m-1}{n} = c,$$

hence

$$P^m(1-P)^{n-1}n + P^{m-1}(1-P)^n m = \frac{n!m!}{(n+m-1)!}c. \tag{18}$$

We refer to the LHS of this equation as $f(P)$. Taking the FOC yields

$$P^{m-2}(1-P)^{n-2}[m(m-1)(1-P)^2 - n(n-1)P^2] = 0.$$

This expression has $m-2$ roots at $P=0$ and $n-2$ roots at $P=1$. Define $M \equiv m(m-1)$ and $N \equiv n(n-1)$. Other roots are then given by the equality $(N-M)P^2 + 2MP - M = 0$, hence

$$P = \frac{-M \pm \sqrt{MN}}{(N-M)}.$$

Note that $M < N$, so the denominator is strictly positive. This also implies $M < \sqrt{MN} < N$, or $0 < \sqrt{MN} - M < N - M$, hence the positive root lies strictly between 0 and 1. We refer to this root as $\tilde{P}$. The negative root is strictly smaller than 0. Note that $f(0) = f(1) = 0$, and $f(P) > 0$ for $P \in (0, 1)$. Thus $\tilde{P}$ is a local maximum. Since $P_A = P$ and $P_B = 1 - P$, the probability that B wins the election will be larger than the probability that A wins if $P < \hat{P}$, with $\hat{P}$ some critical value. Define $\hat{c} \equiv f(\min\{\hat{P}, \tilde{P}\}) \cdot (n + m - 1)!/n!m!$ With $f(0) = 0$ and $f' > 0$ on $[0, \tilde{P}]$, using (18), we have an equilibrium with $P \leq \hat{P}$ if $c \leq \hat{c}$. This establishes Proposition 3.

References

de Borda JC (1781) Memoire sur les elections au scrutin. Histoire de l'Academie Royale des Sciences, Paris

de Condorcet M (1785) Essai sur l'application de l'analyse à la probabilité des voix. De l'imprimerie royale, Paris Translated as Essay on the application of mathematics to the theory of decision making. In: Baker K (ed) Condorcet, selected writings. Bobbs-Merill, Indianapolis (1976)

Myerson RB (2000) Large poisson games. J Econ Theory 94: 7–45

Nurmi H (1998) Voting paradoxes and referenda. Soc Choice Welfare 15: 333–350

Osborne MJ, Rosenthal JS, Turner MA (2000) Meetings with costly participation. Am Econ Rev 90: 927–943